The Mellin transform in statistics: quadratic forms
distribution and prior specification

1 Brief introduction

The exact distribution function of linear combination of chi squared variables (i.e. a
quadratic form in normal vectors) has been derived by Ruben (1962) in the case of non-
negative weights and then generalized by Provost and Rudiuk (1996) to the indefinite case.
Particular attention has been paid to the ratios of quadratic forms, since they are closely
related to the distribution of several widely used test statistics. Moreover, further research
on the computational side of the problem is required in order to implement efficient algo-
rithms to compute distribution functions of ratios of quadratic forms (Broda and Paolella,
2009).

The study of the distribution of quadratic forms has interesting repercussions in Bayesian
inference. In fact, when a hyperprior on a variance of a random effect having a fixed
dependency structure needs to be specified, such as in the case of intrinsic Gaussian random
Markov fields, the magnitude of the scale might influence the estimation of the model
(Sørbye and Rue, 2014). A possible way to control for it could be based on the marginal
variance of the random effect, whose distribution is related to the quadratic forms.

The proposal of this research is to take advantage of the appealing properties of the
Mellin transform, a mathematical tool already exploited in statistics (Epstein et al., 1948),
in order to fill some gaps present both in quadratic forms distribution literature and in
the variance hyperprior specification procedures when a structured covariance matrix is
considered.

2 Background and statement of the problem

The Mellin transform is strictly connected to the well-known Fourier transform. The Mellin
transform $F(s)$, defined in the complex plane, of a function $f(x)$ defined on the positive
real axis is:

$$F(s) = \int_0^{+\infty} f(x)x^{s-1}dt.$$ 

It represents the complex moment of a positive random variable having density $f(x)$. How-
ever, it is endowed with a plethora of useful properties that makes its use quite flexible.
Among the others, it is worth to point out that the Mellin transform naturally deal with

1
the product of random variables and that the Mellin of a linear combination is simply the linear combination of the Mellin transforms. The inverse Mellin transform is:

\[ f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s) \, ds. \]

The density of a semi-positive definite quadratic form \( y = x^T A x \), where \( x \sim \mathcal{N}(\mu, \Sigma) \), might be expressed as an infinite sum of chi-square densities (Ruben, 1962). The main idea of the work is to deduce the Mellin transform of this distribution, in order to tackle the problem of computing the distribution of a ratio of random variables. Finally, once the Mellin of the ratio is derived, it might be numerically inverted to recover the density or the distribution functions of interest (Abate et al., 2000).

The second part of the project is strictly related to Bayesian inference. It deals with the hyperprior specification for precision terms \( \tau \) in presence of a structured precision matrix \( K \). It is common to fit models including random effects with prior \( u | \tau \sim \mathcal{N}(0, \tau^{-1} K^{-}) \), where \( K \) is fixed according to a particular structure: in this case the scale determined by \( K \) affects the model fitting and choosing the same hyperprior for precisions related to structured effects having different scales might represent a non-optimal procedure. This problem has a long tradition in the Bayesian literature (Bernardinelli et al., 1995; Wakefield, 2006), and, particularly, Sørbye and Rue (2014) proposed to standardize the scales by dividing the variance term with a reference variance, that is fixed equal to the geometric mean of the elements on the diagonal of the generalized inverse of \( K \). However, this choice is motivated mainly by heuristic arguments and a more rigorous procedure might be investigated. Therefore, in order to quantify the variance of the random effect \( u \), it is necessary to study the quantity \( u^T M u \), which is a quadratic form, conditioned with respect to \( \tau \), and with a centering matrix \( M \). As a consequence, the marginal distribution of the variance of \( u \) is \( \int p(u^T M u | \tau) p(\tau) \, d\tau \).

### 3 Research question or hypothesis, aim, objectives and deliveries

In the first branch of the research project, the main topic concerns the quadratic forms distributions and the use of the Mellin transform to find new procedures to compute them. The main key points might be summarized as follows.

- A primary aim of the research consists of producing an efficient algorithm for the computation of the distribution of ratios of quadratic forms having the matrix \( A \) as general as possible. This might lead to remarkable improvements in the computation of the distributions of several test statistics like Durbin-Watson or Moran’s \( I \).

- Another contribution of the research is expected in the special function literature. From the results by Fleiss (1971) about the distribution of the linear combination of chi-squared variables, it is possible to identify a strict connection with the Carlson’s \( R \)
function (Carlson, 1963), that is strictly connected to the Lauricella’s hypergeometric function $F_D$. The $R$ function is used in statistics because of its relationship with the Dirichlet distribution (Dickey, 1983). In this case the aim is to provide new computational strategies based on the inversion of a Mellin transform.

As hinted in the previous sections, the quadratic forms play an important role in the hyperprior specification when structured precision matrices are specified in random effects priors. In this case, the main objective for this branch of research is the following one.

- To control the prior information specified in presence of structured random effects the marginal distribution of the random effects variance should be considered. They involve the product of two random variables, one of which is a quadratic form: in this way the use of the Mellin transform developed in the first part of the research might help in producing rules that allows to consciously specify the priors of the precision terms.

Finally, in order to encourage the diffusion and the accessibility of the developed methodologies, a crucial part of the work should involve the construction of $R$ packages: one about the distributions of quadratic forms and and one about the prior specification. The speed of implemented algorithms might be increased using the C++ language.

The research activity will follow the plan reported below:

- Survey and synthesis of the scientific literature (month 1);
- Theoretical developments focused on the Mellin transform for quadratic forms and ratios of quadratic forms; computation of the Carlson $R$ function and Lauricella $F_D$ (months 2-7);
- Developments of prior specification in Bayesian random effect models: solution of integral equations (months 5-9);
- Software implementation, development of $R$ packages (months 6-12);

The fellow is expected to deliver at least two working papers and to disseminate the advancements of his research in two international conferences.

4 Participants in the study and the role they play

The research will involve two members of the Department of Statistical Sciences:

- Daniele Ritelli will be involved as a specialist in the field of special functions
- Fedele Greco will be involved as a specialist in the field of prior elicitation in Bayesian random effect models
References


